

This homework is due November 10, 2014, at 12:00 noon.

1. Section Rollcall!

In your self-grading for this question, give yourself a 10, and write down what you wrote for parts (a) and (b) below as a comment. Please put the answers in your written homework as well.

- (a) What discussion did you attend on Monday last week? If you did not attend section on that day, please tell us why.
- (b) What discussion did you attend on Wednesday last week? If you did not attend section on that day, please tell us why.

2. Biased Coins Lab

While waiting to hear back from the competitively elegant CS (Charm School) program at Cal, you decide to toss coins and plot the result to calm your nerves. Denote p as the probability of tossing a head, and $1 - p$ as the probability of tossing a tail. In the previous two labs, p was 0.5, since we were exploring fair coin tosses. Now assume p can be any number such that $0 \leq p \leq 1$. As before, let k denote the total number of coin tosses.

At an abstract level, most of what this lab is about is doing the previous labs again, except that now the coin is biased. One new concept is introduced: we now need to understand how the bias of the coin affects the shapes of curves that emerge.

For each part, students who want to can choose to completely rewrite the question. Basically, you can come up with your own formulation of how to do a series of experiments that result in the same discoveries. Then, write up the results nicely using plots as appropriate to show what you observed. You can also rewrite the entire lab to take a different path through as long as they convey the key insights aimed at in each part.

Please download the IPython starter code from Piazza or the course webpage, and answer the following questions.

- (a) Consider $p = 0.7$, $\bar{p} = 1 - p = 0.3$. If you plotted a histogram for the number of heads and tails for any number of coin tosses k , what would you expect to see that is different from the fair coin toss?
- (b) If you tossed 100 coins, approximately how many heads should you get? Building on this, can you come up with an equation using $k, p, \bar{p}$ that approximately describes how many heads and tails you expect to see for any case?
- (c) Use a random number generator to sample a sequence of coin tosses with $p = 0.7$, $\bar{p} = 0.3$. Plot a bar chart for each $k = 10, 100, 1000, 4000$. Is this what you expected from parts (a) and (b)? How well does your equation match for $k = 10$? For $k = 4000$?

Hint: Implement the functions `biased_coin`, which simulates a biased coin flip with $\Pr(\text{Head}) = p$, and `run_trial`, which returns the number of heads in k tosses of a biased coin with probability p of getting heads. They should be very similar to the ones you implemented previously for the fair coin.

- (d) From the previous parts, you've counted the number of heads and tails for only one trial. Now fix k to be 1000. Let S_k be the total number of heads in a trial with k total coin tosses, and let m be the total number of trials.

Plot a histogram of S_k for $m = 1000$ with bin size of 1. Do this again for $k = 10, 100, 4000$.

Hint: Implement the function `run_many_trials`, which returns a list of the number of heads in each trial.

- (e) Repeat the previous part, but instead plot the histograms of $\frac{S_k}{k}$ with bin size of 0.01. Make sure to plot the histograms in the same figure (so only one plot) for different values of k . How are the histograms different as k increases? Comment on what you are observing.

- (f) In last week's lab, we discovered a very interesting normalization by $\sqrt{k}$ that seemed to make certain curves fall right on top of each other. Let's see if that still works.

Redo the plot you did in part (g) of last week's lab, except for our biased coin and the k values you have explored above. Center the horizontal axis of the plot to be around 0.

- (g) For kicks, let's try this same normalization for histograms. Plot histograms of $\frac{S_k - kp}{\sqrt{k}}$ for $m = 1000$ and $k = 10, 100, 1000, 4000$. Try 4 different bin sizes, 0.01, 0.05, 0.1, 0.5. You should have 4 different plots, one for each bin size.

Comment on what you think the relationship is between these histograms and the "cliff-face" curves in the previous part. Think about what they mean.

- (h) Now, let's see what the effect is of varying the bias of the coin. Repeat the previous two parts for $p = 0.1, 0.3, 0.5, 0.9$. You can assume the bin size is 0.1. You should have 4 pairs of plots.

Make sure that all of these are plotted on the same scale as the previous parts. What do you observe? Which p seem more variable even after this $\sqrt{k}$ normalization? Less variable?

- (i) Based on what you observe in the previous pattern, you decide to try and hunt out the actual dependence on the shape by looking at appropriate plots. As you did in part (f) of last week's lab, plot the Log of the gap between the 0.75 and 0.25 quartiles against $\ln p + \ln(1 - p)$ in a scatter plot for $k = 4000$. (For this, you could also just plot the gap against $p(1 - p)$ on a Log-Log plot.) What do you observe?

Hint: Implement the method `calculate_quartile_gap`, which calculates the gap between the first and third quartile.

You can also plot a scatter plot in Matplotlib with `plt.scatter()`.

- (j) This suggests another normalization. We call $\sqrt{p(1 - p)}$ the "standard deviation" for a 0-1 coin toss with probability p of being 1.

Plot a histogram of $\frac{S_k - kp}{\sqrt{k}\sqrt{p(1-p)}}$ for $k = 1000$ with bin size of 0.1. Do this again for $k = 10, 100, 4000$ and for $p = 0.1, 0.3, 0.5, 0.7, 0.9$ as before. You should have one plot for each value of p , so there should be 5 plots in total.

What do you observe?

- (k) Use the same normalization as the previous part and make the cliff-face plots corresponding to part (d) of last week's lab. To be precise, look at the range $d = -3$ to $d = +3$ and plot how often (out of the m runs — as a fraction between 0 and 1) $\frac{S_k - kp}{\sqrt{k}\sqrt{p(1-p)}}$ is less than d .

This should be an increasing curve. Here, put all the different p plots together but have three different plots for $k = 100, 1000, 4000$. (You already know from earlier plots that the different k 's track each other.)

- (l) Lastly, we would like to explore a function that defines how many ways you can choose k distinct objects out of n possible objects. This is written as $\binom{n}{k}$ and is read aloud as " n choose k ". We define

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For example, $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)} = \frac{120}{12} = 10$. Plot the value $\binom{50}{k}$ on the y -axis and k on the x -axis for $0 \leq k \leq 50$.

Does this value constantly grow as k gets larger? What does the shape of the graph remind you of?

Hint: Implement the function `choose(n, k)` using `math.factorial`.

Reminder: When you finish, don't forget to convert the notebook to pdf and merge it with your written homework. Please also zip the `ipynb` file and submit it as `hw10.zip`.

3. Charm School Applications

- (a) n males and n females apply to the Elegant Etiquette Charm School (EECS) within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

- (b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The boy(s) get in one car, and the girl(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Each car has a driver...)

4. Getting to CS

Harry, the chosen one, is chosen to drive the boys to the Charm School. The Flying Ford Anglia he's driving is behaving weirdly – it would only go south or east for at least a certain distance. Figure 1 shows the path the car could go from Harry's house (H) to the Charm School (CS).

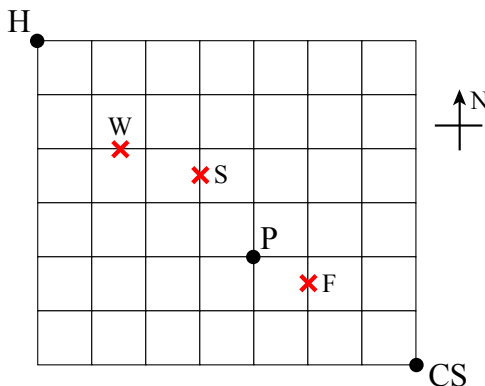


Figure 1: The map from Harry's house to the Charm School

- (a) How many ways can he get there?
- (b) Harry has to pick up other students at point P . How many ways can he stop by point P and go to the Charm School?

- (c) The Whomping Willow (W) will attack anything that comes near. Harry must not drive through it. How many ways can he pick up the students and then go to the Charm School now?
- (d) On top of the Whomping Willow, the Marauder's Map shows Professor Snape (S) and Filch (F) whom he doesn't want to drive past either. How many ways can Harry go to the Charm School without getting past the Whomping Willow (W), Professor Snape (S), or Filch (F), while still picking up other students?

5. Handshakes

In the first day of Charm School, all students are going to practice self-introduction skills. When they meet a new person, they will introduce themselves and shake their hands. Suppose that there are m students who are right-handers and n students who are left-handers. When two people are both right-handers, they will shake their right hands and so will left-handers. When a right-hander meets a left-hander, they will either both shake their right hands or both shake their left hands, with equal probability $p = 0.5$.

- (a) Assume that all students do not know each other at the beginning. How many handshakes will occur?
- (b) Assume that there are k people who have known each other so they will not do any handshake. How many handshakes will occur?
- (c) From now on, we assume all students do not know each other at the beginning. We know that one of the students, Ginny, is a right-hander. What is the probability that Ginny will shake her right hand when she meets the first person?
- (d) If we know that Neville shakes his left-hand in his first handshake, what is the probability that Neville is a left-hander?
- (e) Let x be the number of right-handshakes occurring during the Charm School. What is the possible range of x ?
- (f) If x is within the range you provided in (e), what is the probability that there are exactly x right-handshakes?
- (g) If we randomly pick $y \leq \min\{m, n\}$ students, what is the probability that there are exactly z students who are right-handers? (Assume $0 \leq z \leq y$.)
- (h) Let $m = 5$ and $n = 5$. If we randomly pick $y = 4$ students among them, what is the probability that there are exactly 5 right-handshakes among the 4 students?

6. Seating Arrangements

The Charm School features full course dinner. Everyone is assigned a table. Each table is a round table with ten seats. Each seat has a nametag.

- (a) The staff forgot to put up the name tags on table 1. How many ways can students at that table sit? (Note that ABCDEFGHIJ and BCDEFGHIJA are the same seating arrangement because the table is round, and all seats are identical.)
- (b) Table 2 has name tags but all 10 students there didn't see the name tags and just sat randomly. If all students have different names, what is the probability that all of them have correct name tags?
- (c) What are the chances all the students in table 2 have wrong name tags? (*Hint: Try small cases first and see if you can form a recurrence relation. You might want to use a calculator or ask your friend Wolfram to calculate the final result.*)
- (d) Table 3 has two Freds and three Georges. The rest have distinct names. If all the students there sit randomly, what is the probability that they all have correct name tags?

- (e) At table 10, only the first student arrived didn't see the name tag, so he sat randomly. The 2nd student to arrive sits on her seat if it is free, otherwise she sits on a random seat. The 3rd student to arrive sits on his seat if it is free, otherwise he sits on a random seat. And so on. If all students have different names, what is the probability the last student gets to sit at his/her seat? (*Hint: Again, try small cases first and observe what's going on.*)

7. Dinner Time

Now let's move on to the actual dinner. Each person has all sorts of plates, flatwares, and glasses in front of them, as shown in Figure 2.¹ The basic rule is to start using utensils furthest from your plate and end with the closest ones. Table 1 lists the menu and the corresponding utensils.

Menu	Plates	Flatwares	Glasses
Water	-	-	O
Red wine	-	-	P
White wine	-	-	Q
Bread	K	L	-
Soup	-	J	-
Salad	E	B	-
Fish	F	C, I	-
Meat	G	D, H	-
Dessert	-	M, N	-

Table 1: Courses and utensils



Figure 2: Formal dinner setting

- (a) Ron is confused what utensils to use ('*Wait, I think I'm at the wrong Charm School..*'). Fortunately, he can wait for his server to select the right plates and glasses. He just needs to pick flatwares. All he sees are, 4 forks (B, C, D, and N), 3 knives (H, I, and L), and 2 spoons (J and M). So, for each course served, he mimicks what other people are using. For example, if other people are using a fork and a

¹Source: <http://damoneroberts.tumblr.com/post/51078389219/home-tip-of-the-day-proper-place-setting>

knife, he picks one fork and one knife. (He can't tell the difference between each fork, but can separate forks from knives and spoons just fine.) What is the probability he uses all utensils correctly? Each utensil is collected after each course and can't be used twice.

- (b) Luna just doesn't care. For each course she just picks one or two random flatwares so that all of them are used at the end, and forces the server to serve on one random plate. For each drink she picks a random glass. What is the probability she used at least two things wrong? (If a utensil isn't used in the course it is matched with, then it is used wrongly.)
- (c) (*Optional*) What is the probability Hermione used all correct plates, flatwares, and glasses?

8. Charming Star

At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same.

- (a) How many possible voting combinations are there for the 5 candidates?
- (b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?

9. Write your own problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

10. Bonus

As you have noticed, this homework is themed around the idea of charm school. Bonus points for anyone who comes up with a problem related both to a real-world lesson about etiquette and charm, as well as to counting and/or basic probability. Is there something that you wish your fellow students knew better as far as charm and etiquette goes? Everything is fair game: hygiene, dress, grooming, manners, conversation, politeness, caring, formality, hospitality, open-mindedness, networking, etc. Help make EECS a more charming, welcoming, and gracious environment...