EECS 70 Discrete Mathematics and Probability Theory
Fall 2014 Anant Sahai

Homework 1

This homework is due September 8, 2014, at 12:00 noon.

Administrative details of preparing and submitting homework submissions: In EECS 70 this semester, you will be submitting all homework solutions online. (You might recognize the submission system from EECS 61ABC.) Use your instructional account and follow the same approach you were instructed to use for Homework 1. See your GSI or Piazza for instructions on how to do this.

You are welcome to form small groups (up to four people) to work through the homework, but you **must** write up all your solutions on your own and give credit to the people you worked with or anyone who helped you (this will not diminish the credit you receive).

Although your final submission must be neat (and so typesetting with LaTeXis not a bad option), you are strongly encouraged **not** to directly try to solve the problems in any sort of typesetting environment. Use paper and pencil and scratch paper. Only typeset your solutions after you already pretty-much know exactly what you want to write. Otherwise, you risk wasting a lot of time.

Before submitting, make sure you check carefully that the PDF comes out correctly and correct any errors. We suggest using

```
acroread hw1.pdf
```

Your submission needs to start with the following information:

Your full name
Your login name
The name of the homework assignment (e.g. hw1)
The number of the problem (e.g. 1)
Your section number
Your list of partners for this homework, or "none" if you had no partners

To submit your answers to this homework assignment, create a directory named `hw1`, copy your solution file (hw1.pdf) to that directory, `cd` to that directory, and then give the command

```
submit hw1
```

Note that the file you submit must be called `hw1.pdf`.

1. Virtual Lab

As part of the homework, you will be asked to complete “Virtual Labs" that involve programming and plotting things. Knowing how to simulate and explore what actually can happen is an important skill and is very useful in learning the material and developing an intuition for it. To help students in this process, the course staff has prepared an official EECS 70 virtual machine with a recent version of Ubuntu Linux, Python 2.7, and several libraries we'll be using throughout the class.

For problems that have a significant programming component, we will usually provide IPython Notebook templates that students can use. However, solutions are generally accepted in any programming language that students prefer. The course staff will only support Python/IPython as the official tool to complete the Virtual Labs.

1. Please set up your Virtual Machine according to the instructions on the course web page, available at `http://www-inst.eecs.berkeley.edu/~cs70/fa14/vm/setup.html`
   Once you are done, “answer" this question by writing “I have read the instructions and setup my Virtual Machine accordingly" and signing your name if possible.
2. Start up the Virtual Machine and enter the password as seen on the screen. Launch a terminal (Ctrl + Alt + t), and change into the “cs70" directory (`cd cs70`).
   Type `ipython notebook`. In the browser window that pops up, click on the “python_sample" notebook. This will open up a new tab, which contains one code cell with a few lines of code. Don't worry if you don't understand the code – it will be explained later.
   Finally, either click on Cell and select Run, or click on the “Play" (right triangle) button. Answer this question by telling us what kind of graph or shape you saw (one sentence should suffice).
   I see a bell-shaped curve.
3. Now it's your turn! Try to insert a new cell and type in the code below. Again, don't worry if you don't understand the code, and answer this question by reporting the output (one sentence should suffice).

```
t = np.arange(0.0, 1.0, 0.01)
fig = figure(1)
ax1 = fig.add_subplot(211)
ax1.plot(t, sin(2*pi*t))
ax1.grid(True)
ax1.set_ylim( (-2,2) )
ax1.set_ylabel('1 Hz')
ax1.set_title('What is this magic?')
```

   I see a sinusoidal curve.

Finally, you need to submit this Virtual Lab with your written homework. Inside your VM's terminal, run

```
ipython nbconvert FILENAME.ipynb --to latex --post PDF
--SphinxTransformer.author='YOUR_NAME_HERE'
```

This will create a pdf file named `python_sample.pdf`. Merge this pdf with the pdf containing your solution to the written homework (there are many utilities and online tools to do this, one of which is `http://www.pdfmerge.com/`, and submit **hw1.pdf** as usual. In addition, you need to submit a **hw1.zip** file, which contains your code for this Virtual Lab (“python_sample.ipynb"). We want you to zip your code because later in the course, you may be asked to complete multiple notebooks.

Congratulations, and we hope you enjoyed learning a lot of new stuff in this first homework! As part of the homework, you should start reviewing Python for the upcoming Virtual Labs (EECS 61A is a prerequisite to this class). The course staff will also post some basic review material on Piazza in a few days.

2. Getting started

What is Anant Sahai's second favorite mathematician?

The answer is found on Piazza.

(Why are we having you do this? Piazza is your best source for recent announcements, clarifications on homeworks, and related matters, and we want you to be familiar with how to read the newsgroup.)
The answer is Gauss.

3. Implications: Which are true?

Which of the following statements are true? Briefly explain your answers.

1. If 30 is divisible by 10 then 40 is divisible by 10.
   True. The premise and conclusion are both true.
2. If 30 is divisible by 9 then 40 is divisible by 10.
   True. The premise is false but the conclusion is true.
3. If 30 is divisible by 10 then 40 is divisible by 9.
   False. The premise is true but the conclusion is false.
4. If 30 is divisible by 9 then 40 is divisible by 9.
   True. The premise is false.

4. Karnaugh Maps

Below is the truth table for the boolean function

$$Y = (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge C).$$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

In this question, we will explore a different way of representing a truth table, the *Karnaugh map*. A Karnaugh map is just a grid-like representation of a truth table, but as we will see, the mode of presentation can give more insight. The values inside the squares are copied from the output column of the truth table, so there is one square in the map for every row in the truth table.

Around the edge of the Karnaugh map are the values of the input variables. Note that the sequence of numbers across the top of the map is not in binary sequence, which would be 00, 01, 10, 11. It is instead 00, 01, 11, 10, which is called *Gray code* sequence. Gray code sequence only changes one binary bit as we go from one number to the next in the sequence. That means that adjacent cells will only vary by one bit, or Boolean variable. In other words, *cells sharing common Boolean variables are adjacent*.

For example, here is the Karnaugh map for Y:

		BC = 00	01	11	10
A	0	0	1	0	1
	1	0	1	1	0

The Karnaugh map provides a simple and straight-forward method of minimizing boolean expressions by visual inspection. The technique is to examine the Karnaugh map for any groups of adjacent ones that occur, which can be combined to simplify the expression. Note that "adjacent" here means in the modular sense, so adjacency wraps around the top/bottom and left/right of the Karnaugh map; for example, the top-most cell of a column is adjacent to the bottom-most cell of the column.

For example, the ones in the second column in the Karnaugh map above can be combined because $(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C)$ simplifies to $(\neg B \wedge C)$. Applying this technique to the Karnaugh map (illustrated below), we obtain the following simplified expression for Y:

$$Y = (\neg B \wedge C) \vee (A \wedge C) \vee (\neg A \wedge B \wedge \neg C).$$

		BC = 00	01	11	10
A	0	0	1	0	1
	1	0	1	1	0

1. Write the truth table for the boolean function

$$Z = (\neg A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (\neg A \wedge \neg B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (A \wedge \neg B \wedge C \wedge \neg D).$$

A	B	C	D	Z
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

2. Using your truth table from Part 1, fill in the Karnaugh map for Z below.

CD

AB	00	01	11	10
00				
01				
11				
10				

CD

AB	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

3. Using your Karnaugh map from Part 2, write down a simplified expression for Z.
The four corners can be combined to get

$$Z = \neg B \wedge \neg D.$$

The entire map can be wrapped onto a torus (a donut shape - the way that video games [like Pac-Man or Asteroids] sometimes wrap around so if you move off the right side, you come out the left side, and if you move past the top, you come out the bottom). The ones form a square with only B and D remaining unchanged at 0 and 0 whereas A and C takes on the values $(00, 01, 10, 11)$ which constitutes all possible combinations AC can take.

CD

AB	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

4. Show that this simplification could also be found algebraically by factoring the expression for Z in (1).

$$Z = (\neg A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (\neg A \wedge \neg B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (A \wedge \neg B \wedge C \wedge \neg D)$$

By using the distributive law $(A \wedge B) \vee (A \wedge C) = A \wedge (B \vee C)$, we get the following.

$$Z = (\neg B \wedge \neg D) \wedge ((\neg A \wedge \neg C) \vee (\neg A \wedge C) \vee (A \wedge \neg C) \vee (A \wedge C))$$

As $((\neg A \wedge \neg C) \vee (\neg A \wedge C) \vee (A \wedge \neg C) \vee (A \wedge C))$ is a tautology (fancy word meaning always true no matter what), we get the following simplification.

$$Z = (\neg B \wedge \neg D) \wedge (1)$$

$$Z = (\neg B \wedge \neg D)$$

5. A few proofs

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in the Lecture Notes) you used.

1. For all natural numbers n, if n^2 is even then n^5 is even.

   To show you that you can always use multiple techniques to prove something, we will provide two different methods here.

   First, we will use a direct proof. Assume that n^2 is even. Then there exists an integer k such that $n^2 = 2k$. Then $n^5 = n^2 \times n^3 = 2(kn^3)$ which is twice an integer. Therefore n^5 is also even.

   Second, we will prove this statement by cases. For any generic natural number n there are two cases.

   (a) n is even. In this case n^5 is a multiple of n which is even, so the conclusion of the implication is true. Hence the implication is true.

   (b) n is odd. The product of two odd numbers is odd. Therefore n^2 is odd. This means that the premise of the implication is false. Hence the implication is true.

2. For all natural numbers n, $n^2 - n + 3$ is odd.

   We will prove this statement by cases. For any generic natural number n there are two cases.

   (a) n is even. In this case n^2 is even and n is even. The difference between even numbers is even and therefore $n^2 - n$ is even. The sum of an even number and an odd number is odd, therefore $n^2 - n + 3$ is odd.

   (b) n is odd. In this case n^2 is odd and n is odd. The difference between odd numbers is even and therefore $n^2 - n$ is even. The sum of an even number and an odd number is odd, therefore $n^2 - n + 3$ is odd.

   One can also use a direct proof, i.e., $n^2 - n + 3 = n(n-1) + 3$. Notice that $n(n-1)$ is the product of two consecutive numbers and hence must be even (by an argument parallel to lecture where we showed that the product of three consecutive numbers must be a multiple of 3). The sum of 3 plus an even number must be odd since even plus odd is odd.

3. For all real numbers x, y, if $x + y \geq 20$ then $x \geq 10$ or $y \geq 10$.

   Let $P(i)$ be the predicate "$i \geq 10$" and $Q(j,k)$ be the predicate "$j + k \geq 20$". Then the statement to prove is $\forall (x,y) \in \mathbb{R}^2 \; Q(x,y) \Rightarrow (P(x) \vee P(y))$.

   We will prove this statement by contraposition. The contrapositive of this statement is $\forall (x,y) \in \mathbb{R}^2 \; (\neg P(x) \wedge \neg P(y)) \Rightarrow \neg Q(x,y)$. If the predicate $\neg P(x) \wedge \neg P(y)$ is true, then we have $x < 10$ and $y < 10$. Adding these two inequalities together, we get $x + y < 20$ which means $\neg Q(x,y)$ is true. Hence, since the contrapositive is true, the original implication is true.

4. For all real numbers r, if r is irrational then r^2 is irrational.

   We will disprove this statement by proving its negative. According to the rules of negation for quantifiers, the negative becomes: there exists a natural number r, such that r is irrational and r^2 is rational. To prove an existential statement, it is enough to provide an example, which is essentially a counterexample to the original proposition. This is a direct proof (of the negative), which might also be called a disproof of the original proposition by counterexample.

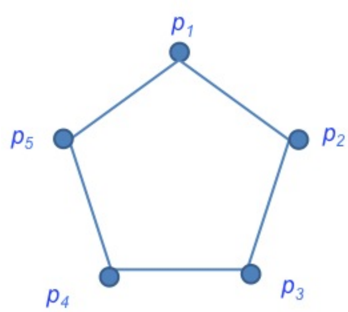
   There are plenty of examples, one of which is $r = \sqrt{2}$. It is well-known that $\sqrt{2}$ is irrational, whereas $r^2 = 2$ is clearly rational.

6. Social Network

Suppose that $p_1, p_2, \ldots, p_n$ denote n people where every two people are either friends or strangers. Let Friends(x, y) be the predicate "x and y are friends". Prove or provide a counterexample for the following statements.

1. For all cases with $n = 5$ people, there exists a group of 3 people that are either all friends or all strangers. In mathematical notation we write this as: $\exists (i, j, k) \in \{1, 2, \ldots, 5\}^3$ such that $i < j < k$ and $(\text{Friends}(p_i, p_j) \wedge \text{Friends}(p_j, p_k) \wedge \text{Friends}(p_i, p_k)) \vee (\neg\text{Friends}(p_i, p_j) \wedge \neg\text{Friends}(p_j, p_k) \wedge \neg\text{Friends}(p_i, p_k))$.

   The statement is false. A counterexample is shown below where people are connected if they are friends and unconnected if they are strangers. In this example, at most 2 are friends or strangers. A group of three people corresponds to three vertices below. Notice that there are only two kinds of groups of 3 vertices. One (like p_1, p_2, p_3) has one person being friends with two people who are strangers to each other. The other (like p_1, p_3, p_4) has two people being friends but both strangers to the third. By the symmetry of this figure, it is clear that these are the only two kinds of groupings.



2. For all cases with $n = 6$ people, there exists a group of 3 people that are either all friends or all strangers. In mathematical notation we write this as: $\exists (i, j, k) \in \{1, 2, \ldots, 6\}^3$ such that and $(\text{Friends}(p_i, p_j) \wedge \text{Friends}(p_j, p_k) \wedge \text{Friends}(p_i, p_k)) \vee (\neg\text{Friends}(p_i, p_j) \wedge \neg\text{Friends}(p_j, p_k) \wedge \neg\text{Friends}(p_i, p_k))$.

   The statement is true.

   Proof: For a person p_1, the rest of the people are either p_1's friends or strangers to p_1, so either p_1's friends or strangers have at least $\lceil \frac{5}{2} \rceil = 3$ people. Why is this true? Suppose they both were less than 3, then there are at most two people that are friends to p_1 and at most two people that are strangers to p_1. $2 + 2 = 4$. But we assumed that everyone is either friends or strangers. So what happened to the fifth person? So we can be sure that at least one of these groups (strictly speaking, exactly one of these groups but we don't need that) has at least 3 people in it.

   Case(1): At least 3 people are p_1's friends

Let $(j,k,l) \in S^3$ where $S = \{2,\ldots,6\}$ such that $j < k < l$ and $(\text{Friends}(p_j,p_1) \wedge \text{Friends}(p_k,p_1) \wedge \text{Friends}(p_l,p_1))$. If p_j,p_k and p_l are all strangers, the statement is true. If not, $\exists(m,n) \in \{j,k,l\}^2$ such that $m < n$ and Friends(p_m,p_n). Then p_m,p_n and p_1 are all friends, so the statement is still true.

Case(2): At least 3 people are strangers to p_i
Let $(j,k,l) \in S^3$ where $S = \{2,\ldots,6\}$ such that $j < k < l$ and $(\neg\text{Friends}(p_j,p_1) \wedge \neg\text{Friends}(p_k,p_1) \wedge \neg\text{Friends}(p_l,p_1))$. If p_j,p_k and p_l are all friends, the statement is true. If not, $\exists(m,n) \in \{j,k,l\}^2$ such that $m < n$ and $\neg\text{Friends}(p_m,p_n)$. Then p_m,p_n and p_1 are all strangers, so the statement is still true.

7. A Weighty Proof

You have 10 bags, each containing 100 coins. Nine of the 10 bags contain genuine gold coins, whereas one bag contains fake coins that are visually indistinguishable from the real gold coins. You don't know which bag has the fake coins, but you do know that real gold coins weigh 10g each while fake ones weigh 10.001g each. You can open the bags, look inside them, take out a few coins, mix them up, etc. You have a weighing machine that you can use *exactly once* – on which you can place a bunch of coins, press a button, and obtain a printed slip showing the weight of the coins placed, down to the milligram.

1. Outline a method to determine which bag has the fake coins.
2. Provide a rigorous proof that your method will indeed always identify the bag with the fake coins, while using the weighing machine exactly once.

The algorithm is as follows:

1. Take 1 coin from the first bag, 2 coins from the second bag, and so on upto 10 coins from the tenth bag. That is, take i coins from the i^{th} bag, for $i \in \{1,2,\ldots 10\}$
2. Weigh these 55 coins using the given weighing scale. Let $x \in \mathbb{R}$ be the resulting weight, in grams.
3. Output the index of the fake bag as $1000(x-550)$.

The correctness of this algorithm is clear, but let us prove it more formally:
We want to show that "the algorithm will output k if and only if the k^{th} bag is fake".

At step (2), we must have $x = 550 + (0.001)k$ where k is the number of fake coins on the scale. Therefore there must be $k = 1000(x-500)$ fake coins on the scale.

By construction, there are k fake coins on the scale if and only if the k^{th} bag was fake. Therefore we output k if and only if the k^{th} bag is fake.

8. Inductions

Prove the following using induction:

1. For all natural numbers $n > 2$, $2^n > 2n+1$.
   1. Base case: The inequality is true for $n = 3$. The Left-Hand Side (LHS) is $2^3 = 8$ and the Right-Hand Side (RHS) is $2(3)+1 = 7$
   2. Inductive hypothesis: Assume the inequality holds for $n = m$: $2^m > 2m+1$.
   3. Inductive step: Prove $2^{m+1} > 2(m+1)+1$

$$\begin{aligned} 2^{(m+1)} &= 2^m + 2^m \\ &> 2m+1+4 \qquad \forall m > 2 \text{ (by hypothesis } 2^m > 2m+1 \text{ and } 2^m > 4) \\ &= 2(m+1)+3 \end{aligned}$$

   Hence, the statement holds by the principle of induction

2. For all positive integers n, $1^3+3^3+5^3+\ldots+(2n-1)^3 = n^2(2n^2-1)$.

   1. Base case: the statement is true for $n=1$. LHS is $1^3=1$ and the RHS is $1^2(2\times 1^2-1)=1$

   2. Inductive hypothesis: Assume that the statement is true for $n=m$.

   3. Inductive step: Prove that it implies that the statement is true for $n=m+1$. That is $1^3+3^3+5^3+\ldots+(2(m+1)-1)^3=(m+1)^2(2(m+1)^2-1)$

$$
\begin{aligned}
1^3+3^3+5^3+\ldots+(2(m+1)-1)^3 &= m^2(2m^2-1)+(2m+1)^3 \text{ (By hypothesis)}\\
&= 2m^4-m^2+8m^3+12m^2+6m+1\\
&= 2m^4+8m^3+11m^2+6m+1\\
(m+1)^2(2(m+1)^2-1) &= 2(m+1)^4-(m+1)^2\\
&= 2m^4+8m^3+12m^2+8m+2-m^2-2m-1\\
&= 2m^4+8m^3+11m^2+6m+1
\end{aligned}
$$

   $1^3+3^3+5^3+\ldots+(2(m+1)-1)^3=(m+1)^2(2(m+1)^2-1)$, so the statement holds for $n=m+1$. Hence, the statement holds by the principle of induction.

3. For all positive integers n, $\frac{5}{4}8^n+3^{3n-1}$ is divisible by 19.

   1. Base case: The statement is true for $n=1$ as $\frac{5}{4}8^1+3^{3\times 1-1}=19$

   2. Inductive hypothesis: Assume that the statement is true for $n=m$: $\frac{5}{4}8^m+3^{3m-1}$ is divisible by 19

   3. Inductive step: Prove that it implies that the statement is true for $n=m+1$. That is $\frac{5}{4}8^{(m+1)}+3^{3(m+1)-1}$ is divisible by 19

   By hypothesis, let $\frac{5}{4}8^m+3^{3m-1}=19k$ where k is some integer

$$
\begin{aligned}
\frac{5}{4}8^{(m+1)}+3^{3(m+1)-1} &= 8*\frac{5}{4}8^m+27*3^{3m-1}\\
&= 8*(\frac{5}{4}8^m+3^{3m-1})+19*3^{3m-1}\\
&= 8*19k+19*3^{3m-1} \text{ by hypothesis}\\
&= 19*(8k+3^{3m-1})
\end{aligned}
$$

   Hence, $\frac{5}{4}8^{(m+1)}+3^{3(m+1)-1}=19l$ where $l=8k+3^{3m-1}$. l is some integer for all $m>0$, so the statement holds for $n=m+1$

   By the principle of induction, the statement holds.

9. Write Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?

10. Registering your EECS 70 instructional account

Answering this question is mandatory to stay in the class. It is also worth a free 4 points.

- To answer this question, log in using your instructional account (cs70-???). Instructional account forms will be handed out in the first discussion sections.
- Register your account using complete and correct information. If for some reason you do not have a student ID number, use your birthday (MM/DD/YYYY) and email Katie at katie.the.headta@gmail.com to notify her.

- Double-check your registration by typing "check-register" and pressing enter.
- Verify that all of the information is correct. Correct any mistakes you find.
- "Answer" this question by writing "I have verified that the registration information for my EECS 70 instructional account cs70-??? is complete and correct" (fill in the ??? with your information) and signing your name if possible.

If your information is incorrect, we will not be able to give you any credit for any of your homework. If you do not know how to do any of the steps above, feel free to get as much help as you need from any source (we suggest Piazza, your classmates, or the readers).