1. Money in a Bag

This problem will give you some practice with variances.

(a) I have a bag full of an equal number of \$1 bills and \$5 bills. If I let you choose a bill uniformly at random from this bag (without charging you a fee), let X be the random variable corresponding to your profit. What is the variance of X?

(b) Say now that I have a new game, in which I do not make any claims about the composition of the bag. However, I do advertise that the **average earnings for one round are \$3**, that **the variance is 3**, and that each round is independent of the previous round.

After playing the game 10 times, your total earnings are \$14–you drew \$1 bills 9 out of 10 times. You suspect me of cheating customers at this game. Can you use expectation and variance to make a case against me? Intuitively the answer is yes–you can show that your earnings are consistently below the advertized expectation, in a way that is more extreme than the advertized variance predicts.

This intuition can be formalized by *Chebyshev's Inequality*. Chebyshev's Inequality states that for a random variable X,

$$\Pr[|X - \mathbf{E}[X]| \geq a] \leq \frac{\operatorname{Var}(X)}{a^2}.$$

i) Define $Y = \sum_{i=1}^{10} \frac{1}{10} Y_i$ to be the random variable corresponding to your average earnings, and let Y_i be the random variable corresponding to your earnings the ith time you play. What is $\mathbf{E}[Y]$, if my claims about the expectation and variance are true?

ii) What is $\operatorname{Var}(Y)$, if my claims are correct? Recall your answer from problem 1c, and that each time you play is independent of the others.

iii) Let $|1.40 - \mathbf{E}[Y]| = a$. Give an upper bound on the probability that $Y = \$1.40$ (if my claims about the expectation and variance are true), by using Chebyshev's Inequality to upper bound the probability that $|Y - \mathbf{E}[Y]| \geq a$.

iv) Say you decided that this probability is not convincing enough. You decided to repeat the game 100 times, and after these 100 times you find that your average earnings are still \$1.40.
Define a new random variable corresponding to your average earnings, $Z = \sum_{i=1}^{100} \frac{1}{100} Z_i$, where Z_i is your earnings on the ith round. Repeat the process above (calculate $\mathrm{Var}(Z)$, then use Chebyshev's Inequality) to give the best upper bound you can on the probability that $Z = \$1.40$.

v) Even though the average is the same after 10 and 100 games, do you have a stronger case for my cheating after repeating the experiment more times?