1. The Central Limit Theorem.
Let us return to a previous online homework problem: recall the game in which I have a bag full of \$1 and \$5 bills, and each round you draw a single bill. I advertise that the average profit of a single round is \$3, and that the variance is also 3 (Note: $\sqrt{3} \approx 1.732$).

a) Suppose you play 10 rounds and have an average profit A_{10} such that $A_{10} \leq \$1.40$. Using the central limit theorem, what is the approximate probability of this outcome for these 10 games?

b) Now, suppose you play 100 rounds and have an average profit A_{100} such that $A_{100} \leq \$1.40$. Using the central limit theorem, what is the order of magnitude of the probability of this outcome for these 100 games?

2. Chebyshev and Chernoff Bounds
Consider a biased coin with probability $p = 1/3$ of landing heads and probability $2/3$ of landing tails. Suppose the coin is flipped some number n of times, and let X_i be a random variable denoting the i^{th} flip, where $X_i = 1$ means heads, and $X_i = 0$ means tails. Let random variable $X = \frac{1}{n}\sum_{i=1}^{n} X_i$. Compute the following expectation and varance:

a) What is $\mathbf{E}[X_i]$?

b) What is $Var[X_i]$?

c) What is $\mathbf{E}[X]$?

d) What is $Var[X]$?

Now we try to use both the Chebyshev's Inequality and the Chernoff Bound to determine a value for n so that the probability that more than half of the coin flips come out heads is less than 0.001.

e) The Chebyshev's Inequality says that for a random variable X with expectation $\mathbf{E}[X] = \mu$, and for any $\alpha > 0$

$$\Pr[|X - \mu| \geq \alpha] \leq \frac{Var[X]}{\alpha^2} \tag{1}$$

According to the definition of probability, we also have

$$\Pr[X - \mu \geq \alpha] \leq \Pr[|X - \mu| \geq \alpha] \tag{2}$$

To determine n, what should α be?

f) What is the minimum value of n according to the Chebyshev Inequality?

g) The Chernoff Inequality says if X_is are i.i.d. and $X = \frac{1}{n}\sum_{i=1}^{n} X_i$, then

$$\Pr[X \geq \alpha] \leq e^{-n\Phi_{X_1}(\alpha)},\ \text{for } \alpha \geq p \text{ or } \Pr[X \leq \alpha] \leq e^{-n\Phi_{X_1}(\alpha)},\ \text{for } \alpha \leq p \tag{3}$$

where $\Phi_{X_1}(\alpha)$ is called the Kullback-Liebler Divergence, usually denoted by $D(a||p)$

$$D(a||p) = \Phi_{X_1}(\alpha) = \alpha \ln \frac{\alpha}{p} + (1-\alpha)\ln \frac{1-\alpha}{1-p} \tag{4}$$

To determine n, what should α be?

h) What is the minimum value of n according to the Chernoff Bound?