1. **Asymmetric Division** True or False? (Prove or give a counterexample):
For an arbitrary random variable X,

$$\Pr[|X-\mathbf{E}[X]| \geq \alpha] < k \implies \Pr[X-\mathbf{E}[X] \geq \alpha] < k/2$$

That is, can we convert a two-sided bound (like Chebyshev's) into a tighter one-sided bound by dividing by two?

2. **Chebyshev's Inequality vs. Central Limit Theorem**
Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the following distribution:

$$\Pr[X_i = -1] = 1/12; \quad \Pr[X_i = 1] = 9/12; \quad \Pr[X_i = 2] = 2/12.$$

(a) What are the expectations and variances of X_i, $(\sum_{i=1}^n X_i)$, $((\sum_{i=1}^n X_i) - n)$, and $\left(\frac{(\sum_{i=1}^n X_i)-n}{\sqrt{n/2}}\right)$?

(b) If $Z = \frac{(\sum_{i=1}^n X_i)-n}{\sqrt{n/2}}$, use Chebyshev's Inequality to find an upper bound b for $\Pr[|Z| \geq 2]$.

(c) Can you use b to bound $\Pr[Z \geq 2]$ and $\Pr[Z \leq -2]$?

(d) As $n \to \infty$, what is the distribution of Z?

(e) A normal distribution N has the following property: $\Pr[|N-\mu| \leq 2\sigma] \approx 0.9545$. As $n \to \infty$, can you provide approximations for $\Pr[Z \geq 2]$ and $\Pr[Z \leq -2]$?

3. Erasures, Bounds, and Probabilities

You may want to use a calculator for this problem. Better yet, use a computer running MATLAB or Python to calculate things like `erf` functions. Or you can just type stuff like `erf(0.25)` into Wolfram Alpha (`http://www.wolframalpha.com`). Or you can use the table on the next page.

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is p, and the erasure of each bit is independent of the others. Alice is using a scheme that can tolerate up to one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most 10^{-6}.

(a) Use Markov's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

(b) Use Chebyshev's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}. Recall that the variance of the number of erased bits will be $1000p(1-p)$, as proved in a previous discussion section.

(c) Use the following Chernoff bound (which is valid for all $\varepsilon > 0$) to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

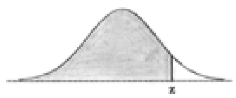
$$\Pr\left(\text{No. of erasures } \geq \ (1+\varepsilon)1000p\right) \leq e^{-\frac{\varepsilon^2}{2+\varepsilon}1000p} \quad \text{(Chernoff Bound)}$$

(d) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable[1] (with suitable mean and variance). Use this to find an approximate bound for p such that the probability of a communications breakdown is at most 10^{-6}. You can use the fact that if Y is a Gaussian random variable with mean μ and variance σ^2, then:

$$\Pr\left(Y \geq y\right) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{y-\mu}{\sigma\sqrt{2}}\right)\right)$$

(e) Describe how you could find the actual maximum p, using a computer. (If time allows, try this and see how it compares).

[1] A Gaussian random variable is an example of a continuous random variable, which you have not seen too much of in this course. Basically, the random variable can be any real number, instead of being confined to take only one of a discrete set of values. For continuous random variables like the Gaussian random variable, it does not make much sense to ask "what is the probability of the variable assuming a particular value, like 0 or 0.5" (since this probability is always 0 for any such single point). Instead, one asks "what is the probability that the variable takes a value in a given range, like for example, $[-1,1)$ or $(5,\infty)$. What the CLT says is that when you average a large number of discrete i.i.d Bernoulli random variables, even though the average is still a discrete random variable, the distribution of the average begins to look more and more like that of a continuous (Gaussian) random variable.



Probability Content from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990