EECS 70 Discrete Mathematics and Probability Theory

Spring 2013 Anant Sahai

Final exam

Section 1: Mandatory straightforward questions (50%)

You get one drop: do 9 out of the following 10 questions.

1. **Find The Inverse**

   Find the multiplicative inverse of 3 mod 28.

   (Show your work)

2. **Induction**

   Prove by induction that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

3. **Polling**

   We don't know what is the probability p of students in 70 liking hard questions involving Orpheus and chickens. How many people should we randomly poll to get a moderately reliable estimate (80% confidence) of the percentage (within $\pm 5\%$) of the CS70 student population that enjoys doing hard exam problems involving Orpheus and chickens?

   (Show your work)

4. **This exam**

   The CS70 prof makes exam questions hard with probability 0.8 and easy with probability 0.2. The Head GSI thinks a hard question is easy with probability 0.4 and think that an easy question is hard with probability 0.2. To protect the students, the Head GSI rejects all final exam questions that she thinks are hard. Only the questions she thinks are easy make it onto the exam.

   What is the probability that a question on the exam is hard?

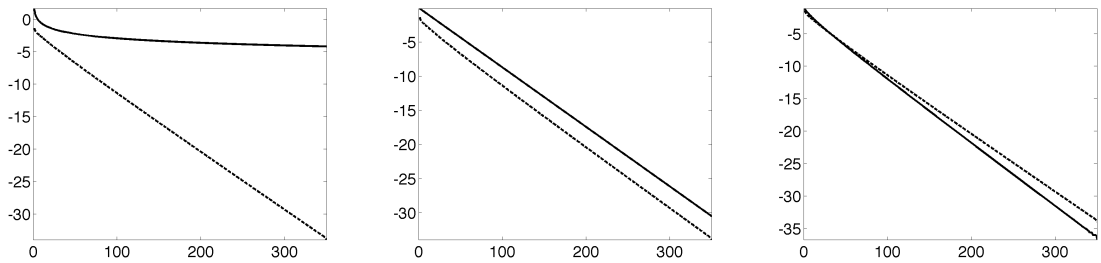
   (Show at least some work)

5. **Plot matching**

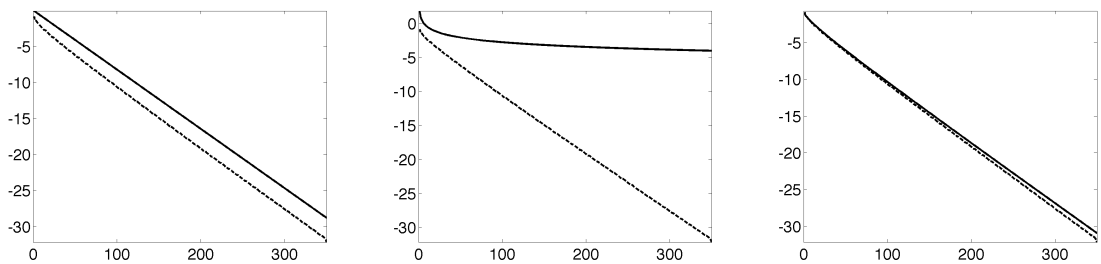
   We have $n = 1$ to 350 Bernoulli-p random variables. Let A be the average of these random variables. The following plots are the logs of various bounds and approximations plotted for the probability of the average being more than 0.2 plus the mean of A. The dashed line represents the actual probability of the deviation (albeit smoothed).

   In each of the cases below, clearly label which one corresponds to the Central Limit Theorem, to Chebyshev's inequality, and to a Chernoff bound.

1. In these plots, $p = 0.3$:



2. In the next set of plots $p = 0.5$:



6. Deja vu?

How many strings of length $4n$ have exactly n letter 'a's, n letter 'b's, n letter 'c's, and n letter 'd's? Use Stirling's approximation to estimate how fast this grows as a function of n.

7. Go Cardinals!

For each of the following collections, determine whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

No explanations are required, just circle the correct choice. Incorrect answers will be penalized. *However, only 5 out of 7 correct answers are required for full credit.*

1. The integers which divide 8:
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
2. The integers which 8 divides
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
3. The alphanumeric strings of length 10
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
4. The pairs (n, s) where n is a natural number and s is an alphanumeric string of length n
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
5. The functions from $\mathbb{N}$ to $\mathbb{N}$
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
6. The functions from $\{0, 1, 2, 3, 4, 5\}$ to $\mathbb{Q}$
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
7. Prime numbers
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE

8. QUESTION DELETED BECAUSE IT HAS TO DO WITH CONTINUOUS PROBABILITY

9. Decode This

The following pairs of points were received while using a standard polynomial-based Reed-Solomon code of length 4 within GF(5) that is designed to withstand 1 malicious error. You know that it is possible to correctly decode. **Please identify whether any points have been corrupted and give the true underlying polynomial**.

$$(0,1),(1,2),(2,4),(3,4)$$

(Show your work or give some explanation as to how you got to your answer and why you believe it to be correct.)

10. Stable marriage + Probability

Suppose you have two men (A and B) and two women (1 and 2). The men's preferences between the two women are drawn independently and uniformly at random (i.e. each man is equally likely to prefer 1 to 2 or 2 to 1); each woman's preference is also drawn independently and uniformly at random (i.e. each woman is equally likely to prefer A to B or B to A). **What is the probability that the male-optimal stable matching is different from the female-optimal stable matching?** Show your work.

Section 2: True/false (15%)

For the questions in this section, determine whether the statement is true or false. If true, prove the statement is true. If false, provide a counterexample.

You get one drop: do 3 out of the following 4 questions.

11. If $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$, then $P(A) > P(B)$. (Remember: C^c is the complement of C.)

Mark one: TRUE or FALSE.

12. A degree-d polynomial is uniquely determined by d points.

Mark one: TRUE or FALSE.

13. If p is a prime greater than 7, then 9 is the multiplicative inverse of 3^{p-3} in GF(p).

Mark one: TRUE or FALSE.

14. If $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$, then X and Y are independent.

Mark one: TRUE or FALSE.

Section 3: Harder Story Problems (35%)

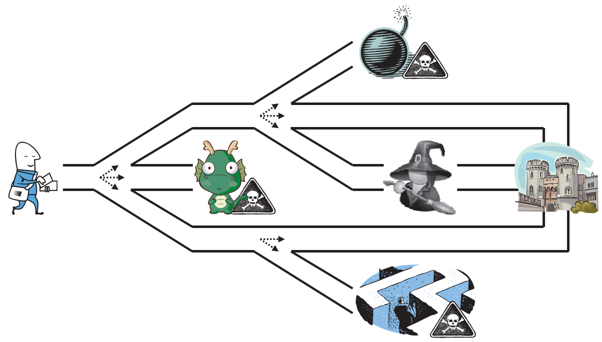
15. Berkeley and Stanford Prove the following combinatorial identity by any means necessary (Hint: a story involving alumni from two schools is sufficient):

$$\sum_{k=1}^{n} k \binom{b}{k} \binom{s}{n-k} = b \binom{b+s-1}{n-1}$$

$(n < b$ and $n < s)$

16. Unreliable Messengers

You are trying to send a message to your base by means of courier.



a) Unfortunately, your couriers don't know the way to the base (depicted as the castle in the figure). As they come to any fork in the road, they just pick a path to take uniformly at random. Unfortunately, three of the paths lead to the death of the courier. And another path leads past a powerful sorceress who will maliciously alter whatever the courier is going to report. **Assuming that 180 couriers were sent toward the castle, what is the expected number of them that arrive with their message intact? What is the expected number of them that will arrive with their message corrupted?**

b) For this part, assume that the probability of a courier being maliciously corrupted by the sorceress is $\frac{1}{6}$ and the probability of a courier being lost is $\frac{1}{3}$. *Note: these probabilities do not correspond to the picture in part (a).*

Each courier can carry a single number from a finite field of size F. (Assume F is quite large.)

You decide to use a Reed-Solomon code to protect what you want to say. Assume that you have 10 numbers' worth of information that you want to encode and reliably transmit.

You would like the probability of correct decoding to be about 95%. How many couriers should you send out?

c) In your scheme of the previous part, what is the approximate probability that the sorceress will see enough couriers to be able to decode the message herself? Assume for this part only that she didn't know it already. Also assume that the probabilities are as given in part b, not what you calculated in part a.

17. QUESTION DELETED BECAUSE IT HAS TO DO WITH CONTINUOUS PROBABILITY

Section 4: Optional extra credit question

We suggest doing this only if you have time: it won't be worth that *much.*

18. Chicken Run Alas, Orpheus' chicken container is busted. So n escaped chickens run (one at a time) past an infinite gauntlet (series) of guards. Each guard independently has a probability $\frac{1}{n}$ to pick up and capture a chicken that tries to run past that guard. (As you can clearly see, every chicken is eventually going to get picked up.)

Let M_n be the random variable for the maximum number of chickens captured by a guard, when n chickens try to escape in this manner. Let t_n be that value for which $P(M_n \geq t_n) = \frac{1}{2}$. Find the best upper bound on t_n that you can.