EECS 70 Discrete Mathematics and Probability Theory
Spring 2014 Anant Sahai Final exam

PRINT your student ID: | | | | | | | | |

PRINT AND SIGN your name: ____________________, ____________________ _________________________
(last) (first) (signature)

PRINT your Unix account login: cs70-______________

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

Name of the person to your right: ____

Name of someone in front of you: ____

Name of someone behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What is your favorite thing about math/EECS? (1 pt)**

2. **What are you most looking forward to this summer? (2pts)**

Do not turn this page until the proctor tells you to do so.

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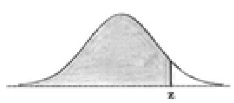
SOME APPROXIMATIONS AND OTHER USEFUL TRICKS THAT MAY OR MAY NOT COME IN HANDY:

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n \qquad \binom{n}{k} \leq (\frac{ne}{k})^k \qquad \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$$

When x is small, $\ln(1+x) \approx x$

When x is small, $(1+x)^n \approx 1+nx$

The Golden Rule of 70 (and Engineering generally) applies: if you can't solve the problem in front of you, state and solve a simpler one that captures at least some of its essence. You might get partial credit for doing so, and maybe you'll find yourself on a path to the solution.



Probability Content from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Source: `http://www.math.unb.ca/~knight/utility/NormTble.htm`

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Section 1: Straightforward questions (24 points)

You get one drop: do 4 out of the following 5 questions. Bonus for getting all five perfectly. No partial credit will be given.

3. Birthday

What is the probability that at least two of your EECS70 classmates (there are 530 of you all) have the same birthday (years don't matter; only month and day do)?

4. Stirling

Use Stirling's approximation to estimate the probability of flipping 50 fair coins and getting exactly 25 heads. Is this bigger or smaller than 0.1?

5. RSA

Let $p = 11$, $q = 17$ in standard RSA. Let $e = 3$.

a) Compute the private key that Bob should use.

b) Let $x = 20$ be the message that Alice wants to send to Bob. **What is the encrypted message** $E(x)$?

6. Count It

For each of the following collections, determine whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

No explanations are required, just circle the correct choice. *Only 6 out of 9 correct answers are required for full credit.*

1. Even numbers:
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
2. The number of distinct files that are 1GB in length
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
3. Computer programs that halt
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
4. The functions from $\mathbb{N}$ to $\{0,1,2,3\}$
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
5. The number of points in the unit square $[0,1]\times[0,1]$
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
6. The functions from $\mathbb{R}$ to $\{0,1,2,3,4,5\}$
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
7. Computer programs that always correctly tell if a program halts or not
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
8. Numbers that are the roots of nonzero polynomials with integer coefficiencts
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE
9. Computer programs that correctly return the product of their two integer arguments
   Circle one: FINITE / COUNTABLY INFINITE / UNCOUNTABLY INFINITE

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7. Induction

Prove by induction that for every natural number $n \geq 1$, we have:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 2: Additional straightforward questions (40 points)

You get two drops: do 4 out of the following 6 questions. Bonus for getting more than that perfectly. Very little partial credit will be given.

8. Frying Pan

Rapunzel and Flynn decide to get Elsa a special gift for her coronation. They decide to order it from Oaken's Trading Company.

Pretend there are just 3 warehouses, one in Corona, one in Wesselton, and one in Arendelle. When someone orders a product, the Corona warehouse is first checked to see if the product is in stock there. The probability that the product is in stock here is 50%. If it is in stock, it is shipped for delivery in either 1, 2, or 3 weeks, all equally likely.

If the product is not in stock in Corona, the next closest warehouse (Wesselton) is checked, which has an 80% probability of having the product in stock. Again, if the product is in stock here, it is shipped, but this time, the product will arrive in 2, 3, or 4 weeks, all equally likely.

If the product is not in stock in Wesselton either, it will be shipped from Arendelle (which is assumed to have all tradeable goods), and the delivery will be made in either 3 weeks (with 10% probability) or 4 weeks (with 90% probability).

Rapunzel ordered a frying pan ("who knew!") from Oaken's Trading Company, and it was delivered in 3 weeks. What are the chances that it was shipped from Arendelle?

9. Combinatorial identity

Use a combinatorial argument to prove why the following combinatorial identity is true (*i.e.*, provide a story for why the identity should hold).

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

10. Doubles

You are playing a game of chance where the rules are as follows. You start out with \$1. You roll a pair of independent and fair 6-faced dice. If you roll "doubles" (both dice agree on a single number), you double your money and get to roll the dice again. If the two dice are different, the game is over and you walk away with your money. Basically: you keep rolling the dice and doubling your money until you don't roll doubles.

What is the expectation and variance of the final money at the end of the game?

11. Seating Senators

Assume there are 100 galactic senators, each one from a different star system, to be seated in a 100-seat row (not a circle. Seats 1 and 100 are not next to each other) in the galactic senate. *You can leave your answers as expressions involving factorials and choose notation.*:

a) In how many ways can the senators be seated, assuming any senator can be assigned any seat?

b) In how many of the ways above will the senator from Naboo and the senator from the Trade Federation be seated next to each other?

c) In how many of the ways will the senators from Naboo and the Trade Federation be seated next to each other *while* the senators from Alderaan and the Techno Union are also seated next to each other?

d) Use the inclusion exclusion principle, along with your answers to the sub-problems above, to **find the probability that, when a seating arrangement is picked uniformly at random, neither the senators from Naboo and the Trade Federation, *nor* the senators from Alderaan and the Techno Union, are seated next to each other.** (No need to simplify your expression. You can just use A for the answer to part (a), B for the answer to part (b), and C for the answer to part (c) to avoid having to rewrite those expressions.)

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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12. No Soulmates

Here are some partial preferences for 4 men and 4 women. **Fill in the missing preferences so that there are** 4 **different stable pairings possible.** Give a brief justification for why there are 4 different stable pairings.

Man	Women			
1	A	B	C	D
2	B			D
3	C	D	A	B
4	D			B

Woman	Men			
A				4
B				4
C				2
D				2

13. Time Sink

In Fall, every EECS70 student (assume there are $625 = 25^2$ of them) will be asked to do a supposedly easy virtual lab in the second week of class. Let X_i be the number of hours it takes for student i to complete the lab. Assume that the $\{X_i\}$ are independent geometric random variables each with expectation $\frac{1}{p}$ and variance $\frac{1-p}{p^2}$.

a) If the *average* time for students to complete the first virtual lab is greater than 4 hours, the morale in EECS70 will crash. **Use the central limit theorem to find an approximate value for p (i.e. the lab difficulty parameter) such that the probability of a decline in morale is about 15.87%.**

(It is fine to leave the answer in the form of a solution to an equation.)

b) BONUS: What is the approximate distribution for the number of students who will spend more than 10 hours on the lab?)

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 3: True/False (42 points)

For each question in this section, determine whether the given statement is TRUE or FALSE. If TRUE, prove the statement. If FALSE, provide a counterexample.

14. Variance

If $n > 1$ and $\mathrm{Var}(\sum_{i=1}^{n} X_i) = n\mathrm{Var}(X_1)$, then the $\{X_i\}$ are iid.

Mark one: TRUE or FALSE.

15. Déjà Vu?

Let $n \geq 1$ be a positive integer. Let $r \geq 1$ be a positive integer. Consider a polynomial based code in which n character messages are encoded into polynomials of degree less than or equal to $n-1$. The codewords are generated by evaluating these polynomials at $n+r$ distinct points (assume the underlying finite field has more than $n+r$ elements).

Then any two codewords corresponding to two different messages must differ in at least $r+1$ places.

Mark one: TRUE or FALSE.

16. Bound

Let X be a random variable with expectation μ_X and variance σ_X^2. Suppose $\mu_X \leq \mu_{\text{bound}}$, and $\sigma_X^2 \leq \sigma_{\text{bound}}^2$. Then, for every $a \geq \mu_{\text{bound}}$, we have:

$$\Pr(X \geq a) \leq \frac{\sigma_{\text{bound}}^2}{(a - \mu_{\text{bound}})^2}$$

Mark one: TRUE or FALSE.

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Section 4: Free-form Problems (65 + 35 points)

17. Cathy's Coupons (20 points)

Cathy is trying to collect two coupons. She pays \$1 to get a coupon, but does not know which coupon she will get before paying. There are two stores (A and B) selling the coupons. In store A, the probabilities of getting Coupon 1 and Coupon 2 are both $\frac{1}{2}$. In store B, the probabilities of getting Coupon 1 and Coupon 2 are $\frac{1}{3}$ and $\frac{2}{3}$ respectively.

a) (10 points) If Cathy can pick only one store to buy all her coupons from, which store should she pick so as to minimize her expected cost?

b) (10 points) If Cathy can switch stores at any time, what strategy should she follow to collect both coupons while minimizing her expected cost? Why?

Calculate the expectation of the total cost when following this strategy.

18. Miley's Money (30 + 20 points)

Miley Cyrus decides to get into angel investing in casual gaming startups. The process is simple.

At the start of each quarter, Miley invests some money in one startup and gives them some publicity. At the end of the quarter, Miley sells her stake. If the venture is successful (this happens with 50% probability, independent of other ventures), Miley sells her stake for 10 times her initial investment (paid immediately, in full, in cash). Otherwise, Miley's investment is worthless and she gets back nothing.

Each quarter, Miley invests a fraction $0 \le f \le 1$ of her investment pool in a startup, as described above. The remaining fraction $(1-f)$ is kept as cash and not invested.

So Miley's investment multiplier in quarter k is iid distributed as X_k with probabilities:

$$P(X_k = (1-f) + 10f) = \frac{1}{2}$$

$$P(X_k = (1-f)) = \frac{1}{2}$$

She hires you as an adviser to help her choose her strategy and allocates $B_0 =$\$10M to start investing.

a) (5 points) What is the expectation and variance of X_k?

b) (5 points) Find an expression for B_k — the total value of Miley's investments at the end of quarter k — in terms of B_0 and the random variables $X_1, X_2, \ldots, X_k$.

c) (5 points) Let t be total number of quarters that Miley is going to dedicate to investing/promoting casual games. **What choice of f maximizes $E[B_t]$? What is the resulting expectation?**

d) (5 Bonus points) What is the variance of the resulting B_t in the previous part?

e) (15 points) Miley understands that investing is risky. She is willing to accept a 10% probability of a bad outcome overall. So Miley is interested in a number m_t for which $P(B_t \leq m_t) \approx 0.1$. **Approximate m_t in terms of f assuming that Miley invests for t quarters.** (Feel free to assume that t is large.)

f) (15 Bonus points) **What choice of f maximizes the m_t that you have calculated above? What does f tend to when t gets large?**

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

19. Median (15 + 15 points)

Given a list of numbers with an odd length, the median is obtained by sorting the list and seeing which number occupies the middle position. (e.g. the median of $(1,0,1,0,2,1,2)$ is 1 because the list sorts to $(0,0,1,1,1,2,2)$ and there is a 1 in the middle. Meanwhile, the median of $(1,0,2,0,0,1,0)$ is 0 because the list sorts to $(0,0,0,0,1,1,2)$ and there is a 0 in the middle.)

a) (15 points) Consider an iid sequence of random variables X_i with probability mass function $P_X(0) = \frac{1}{3}$, $P_X(1) = \frac{1}{4}$, $P_X(2) = \frac{5}{12}$. Let M_i be the random variable that is the median of the random list $(X_1, X_2, \ldots, X_{2i+1})$. **Show that $P(M_i \neq 1)$ goes to zero exponentially fast in i.**

(HINT: What has to happen with the $\{X_j\}$ for $M_i = 0$ or $M_i = 2$?)

b) (15 Bonus points) Generalize the argument you have made above to state a law of large numbers for the median of a sequence of iid discrete-valued random variables. Sketch a proof for this law.
(HINT: Let Mid(X) be that value t for which $P(X < t) < \frac{1}{2}$ and $P(X > t) < \frac{1}{2}$. If such a t doesn't exist, then don't worry about that case at all.)

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[Doodle page! Draw us something if you want or give us suggestions or complaints.]