EECS 70 Discrete Mathematics and Probability Theory

Spring 2014 Anant Sahai

Midterm 3

PRINT your student ID: | | | | | | | | |

PRINT AND SIGN your name: ________________, ________________ ____________________

(last) (first) (signature)

PRINT your Unix account login: cs70-____________

PRINT your discussion section and GSI (the one you attend): ______________________________________

Name of the person to your left: ____

Name of the person to your right: ____

Name of someone in front of you: ____

Name of someone behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What is your favorite topic in EECS70 so far? (1 pt)**

2. **Describe a time when you did better than you expected at something? (2pts)**

Do not turn this page until the proctor tells you to do so.

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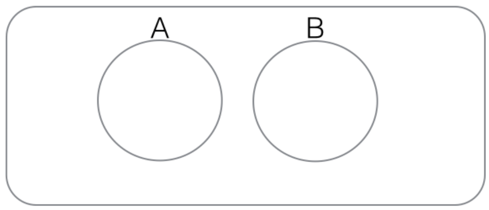
Section 1: Straightforward questions (40 points)

You get three drops: do 4 out of the following 7 questions (we will grade all 7 and keep only the 4 best scores). However, there will be no partial credit given in this section. Students who get 6 or all 7 questions correct will receive some bonus points. You must show work.

3. No Overlap

The rectangle below represents the entire event space. We have depicted two events, A and B as portions of the event space. Notice that events A and B do not overlap. Assume they are not empty.



True or False: Events A and B are independent. Explain your answer.

4. Create a Story

Tell us a story showing why the following combinatorial identity holds. Your proof must not be algebraic (i.e. it cannot invoke an expansion formula for $(a+b)^3$).

$$n^3 = 1 + 3(n-1) + 3(n-1)^2 + (n-1)^3$$

5. Conditional Probability

For each question on a multiple-choice exam, Alice either knows the answer or guesses. Each question has m choices. If she guesses, she chooses one of the m choices uniformly at random. Let p be the probability that she knows the answer. What is the conditional probability that Alice knew the answer to a question, given that she answered it correctly? Express your answer in terms of p and m, but you don't need to simplify.

6. Count

How many solutions of the form $(x_1, x_2, \cdots, x_k)$ are there to the equation

$$x_1 + x_2 + \cdots x_k = 2k,$$

where k is a positive integer and each x_i must be a non-negative integer?

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

7. Distinguishability

How many ways to put 5 balls into 4 bins?

Note that if bins 1 and 2 are indistinguishable, then if balls w and x land in bin 1 and y and z land in bin 2, that is considered *the same* as balls w and x landing in bin 2 and y and z landing in bin 1.

1. If the balls are distinguishable and the bins are distinguishable?

2. If balls are indistinguishable and bins are distinguishable?

3. If balls are indistinguishable and bins are indistinguishable?

8. Hide and Seek

Alice makes 3 identical copies of an important message for the EECS Resistance. She hides these 3 copies in 3 randomly selected (without replacement) rooms out of n previously-agreed-upon classrooms in Cory and Soda Halls. Your task is to determine how large an n the EECS resistance should agree upon in order to secure the message against the UCB Empire.

The UCB Empire has figured out which n classrooms are possibly message hiding places, but doesn't know which 3 rooms were selected by Alice. The Empire sends three operatives, each of which independently and uniformly randomly selects one of the n classrooms to search.

How many classrooms, n, should the EECS Resistance select (before hiding anything) to have less than 50% chance that the message is found by the UCB Empire.

(Numerical Hints: $\frac{1}{\sqrt{2}} \approx 0.7$, $\frac{1}{\sqrt[3]{2}} \approx 0.8$, $\frac{1}{\sqrt[4]{2}} \approx 0.84$*)*

9. Volunteers Deserve Apple-ause

It's apple-picking season, and I have n apples fresh from my orchard that I have numbered 1 to n. I want to thank Alice and Bob for all their hard work in helping me pick apples. The picking took about 10 human-hours total. Alice spent $10c_1$ hours and Bob spent $10c_2$ hours, and I'd like to give them apples proportional to their work. (I keep the rest of the apples for myself.) Here $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$.

I decide to give Alice $k_1 = c_1 n$ apples and Bob $k_2 = c_2 n$ apples. In how many ways can I select apples to give to Alice and Bob? Use Stirling's approximation to show how this number scales exponentially with n.

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Section 2: True/False (30 points)

For the questions in this section, determine whether the statement is true or false. If true, prove the statement is true. If false, provide a counterexample demonstrating that it is false.

10. Busy Office Hours

In a group office hour of CS70, there are n students simultaneously coming to ask questions and there are $m \geq n$ GSIs/readers available. A student in your study group calculates that if each student uniformly randomly goes to any GSI/reader for help independently of what other students are doing, the probability that we don't overwhelm any GSI/reader with more than one student is $\frac{\binom{m}{n}}{m^n}$.

Mark one: TRUE or FALSE.

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11. Choose Math

For any positive integer n,

$$\binom{2n}{n} = 2^n \, \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

Mark one: TRUE or FALSE.

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Section 3: Free-form Problems (30 points)

12. Alladin (15 points)

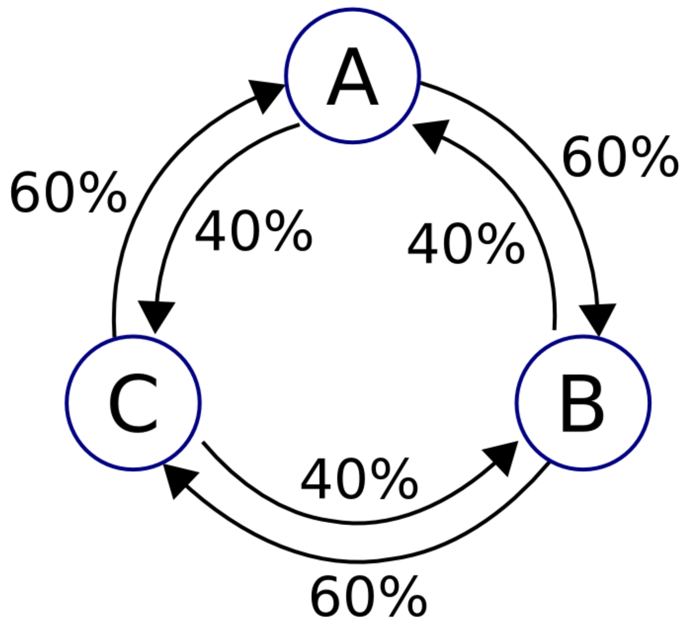

Figure 1: Circular path followed by Alladin.

Imagine that Alladin is stumbling along a circular path in the Cave of Wonders, as shown in the figure above. **Initially (at time $t = 0$), Alladin is in position A.** And at each time point $t \geq 0$, he moves either clockwise, with a 60% probability, or counterclockwise, with a 40% probability. For example, if he is in position B at time 100, there is a 60% chance that he will be in position C at time 101, and a 40% chance that he will be in position A at time 101.

Given that Alladin is in position A at time 3, what is the probability that he was in position A at time 1? Position B at time 1? Position C at time 1?

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13. Djinn and humans (15 points)

Prove the following identity for $n \geq 2$:

$$\sum_{k=1}^{n} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \binom{2n-2}{n-2}.$$

[HINT: A combinatorial proof might be easier. Split the items into two categories. Can you reason about selecting a group of them in some special way?]

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

14. (optional) Safe Hacking (15 points) A hacker is trying to break into the company X's servers and liberate its secrets. The hacker has found a way to make a server reveal all of its stored information by sending a simple request. Unfortunately such a request also makes servers crash after revealing all of their information.

The company has n servers all hidden behind a single access point. Any incoming request goes through the access point. The access point sends any incoming request to a randomly chosen server (uniformly), and returns the results. If a server is unable to answer because it has crashed, the access point again picks a random server **(with replacement)**, and tries again. The access point continues doing this until the request goes through or the number of failed tries for this particular request reaches t, at which point an administrator is notified.

The hacker does not want administrators notified, but wants to liberate the information of as many servers as possible. So the hacker chooses to try liberating k servers' information and then stops.

1. Given that the first $i-1$ attempts went unnoticed (*i.e.* the hacker was successful in breaking $i-1$ servers already), what is the chance that the next attempt also goes unnoticed? Let this be p_i. **Show that for** $i > k/2$**,** $p_i \leq e^{-(\frac{k}{2n})^t}$**.**
   (Hint: use the inequality $1-x \leq e^{-x}$*.)*

2. The previous part gives an upperbound on the chance of remaining unnoticed after k requests. **Prove that for $k \geq (2n)^{\frac{t}{t+1}}$, there is at least a $1 - \frac{1}{\sqrt{e}}$ chance of an admin being notified.**

   So the hacker would be ill-advised to try and liberate the information on more than $(2n)^{\frac{t}{t+1}}$ servers.

   (HINT: First think about the case $t = 1$ to gain intuition if you are stuck and confused about the form of the result.)

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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[Doodle page! Draw us something if you want or give us suggestions or complaints.]